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WAVELETS ASSOCIATED WITH NONUNIFORM MULTIRESOLUTION ANALYSIS ON POSITIVE HALF-LINE

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Gabardo and Nashed have studied nonuniform multiresolution analysis based on the theory of spectral pairs in a series of papers, see Refs. 4 and 5. Farkov,[3] has extended the notion of multiresolution analysis on locally compact Abelian groups and constructed the compactly supported orthogonal p-wavelets on $L^2(\mathbb{R}_+)$. We have considered the nonuniform multiresolution analysis on positive half-line. The associated subspace V_0 of $L^2(\mathbb{R}_+)$ has an orthonormal basis, a collection of translates of the scaling function φ of the form $\{\varphi(x \ominus \lambda)\}_{\lambda \in \Lambda_+}$ where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$, $N > 1$ (an integer) and r is an odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime and $\mathbb{Z}_+$ is the set of non-negative integers. We find the necessary and sufficient condition for the existence of associated wavelets and derive the analogue of Cohen's condition for the nonuniform multiresolution analysis on the positive half-line.

Keywords: Nonuniform multiresolution analysis; Walsh–Fourier transform; wavelets; scaling function.

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1. Introduction

Multiresolution analysis is known as the heart of wavelet theory. The concept of multiresolution analysis provides an elegant tool for the construction of wavelets. A multiresolution on the set of real numbers $\mathbb{R}$, introduced by Mallat,[13] under the inspiration of Y. Meyer is an increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there should exist an element $\phi \in V_0$ such that the collection of integer translates of ϕ, $\{\phi(x-n)\}_{n\in\mathbb{Z}}$ is a complete orthonormal system for V_0. In the definition of multiresolution analysis the dilation factor of two can be replaced by an integer $N \geq 2$ and one can construct $N-1$ wavelets to generate the whole space $L^2(\mathbb{R})$. A similar generalization of multiresolution analysis can be made in higher dimensions by considering matrix dilations, see Ref. 10. Gabardo and Nashed,[6] considered a generalization of the notion of multiresolution analysis, which is called nonuniform multiresolution analysis (NUMRA) and is based on the theory of spectral pairs.

Given a one-periodic trigonometric polynomial m_0 satisfying the identities $m_0(0) = 1$ and $|m_0(\xi + 1/2)|^2 + |m_0(\xi)|^2 = 1$, Cohen's condition provides a necessary and sufficient condition for a function ϕ, defined via the Fourier transform by the infinite product $\hat{\phi}(\xi) = \prod_{k=1}^{\infty} m_0(\frac{\xi}{2^k})$, to be the scaling function of some multiresolution analysis on $\mathbb{R}$ and in particular, for the sequence of translates of ϕ i.e. $\{\phi(x-n)\}_{n\in\mathbb{Z}}$, that generates the associated subspace V_0, to be orthonormal. Gabardo and Nashed,[7] derive the analogue of Cohen's condition for the generalized setting of NUMRA.

Walsh analysis or dyadic harmonic analysis has been extensively studied: both aspects theory as well as applications, see Refs. 3, 6 and 13. Lang,[9–11] initiated the study of wavelet analysis on the Cantor dyadic group. The concept of dyadic multiresolution analysis has been studied on $L^2(\mathbb{R}_+)$, where $\mathbb{R}_+$ denotes the positive half-real line by Protasov and Farkov.[14] Farkov,[3] has given a general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}_+)$ and for all integers $p \geq 2$; these wavelets have been identified with certain lacunary Walsh series on $\mathbb{R}_+$. The approach adopted by Farkov is connected with Walsh–Fourier transform and the elements of M-band wavelet theory. Also Farkov *et al.*,[4] have constructed an algorithm for computing biorthogonal compactly supported dyadic wavelets related to the Walsh functions on the positive half-line $\mathbb{R}_+$. F. A. Shah,[16] studied the construction of p-wavelet packets associated with the multiresolution defined by Farkov,[3] for $L^2(\mathbb{R}_+)$. M. K. Ahmad and J. Iqbal,[1] derived frames and corresponding frame bounds for vector-valued Weyl–Heisenberg wavelets. In the present paper we introduce the notion of nonuniform multiresolution analysis on positive half-real line and study the analogue of the results of Gabardo and Nashed and Cohen's result cited above.

Let us recall the definitions of NUMRA and associated set of wavelets:

Definition 1.1. Given an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, an associated nonuniform multiresolution analysis (abbreviated NUMRA) is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:

(a) $V_j \subset V_{j+1}\ \forall\, j \in \mathbb{Z}$.
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(d) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$.
(e) There exists a function $\phi \in V_0$, called a scaling function such that the collection $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ is a complete orthonormal system for V_0.

It is worth noticing that when $N = 1$, one recovers from definition above the standard definition of a one-dimensional MRA with dyadic dilation. When $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset 2\mathbb{Z} \subset \Lambda$. However, the existence of associated wavelets with the dilation $2N$ and translation set Λ is no longer guaranteed as is the case in the standard setting.

Given a NUMRA, we denote by W_m the orthogonal complement of V_m in V_{m+1}, for any integer m. It is clear from (a), (b) and (c) of Definition 1.1 that

$$L^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} W_m.$$

Definition 1.2. A collection $\{\psi_k\}_{k=1,2,\ldots,2N-1}$ of functions in V_1 will be called a set of wavelets associated with a given NUMRA if the family of functions $\{\psi_k(x-\lambda)\}_{k=1,\ldots,2N-1,\lambda\in\Lambda}$ is an orthonormal system for W_0.

The main results of Gabardo and Nashed deal with necessary and sufficient condition for the existence of associated wavelets, [6, Theorem 3.7] and extension of Cohen's theorem [7, Theorem 4.2]. The following result proved in Ref. 6, provides the simple necessary and sufficient conditions for the existence of the associated set of wavelets.

Theorem 1.1. *Consider a NUMRA with associated parameters N and r, as in Definition* 1.1, *such that the corresponding space V_0 has an orthonormal system of the form $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$, where $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, $\hat{\phi}$ satisfies the scaling relation*

$$\hat{\phi}(2N\xi) = m_0(\xi)\hat{\phi}(\xi),$$

where $\hat{\phi}$ denotes the Fourier transform of a function ϕ and m_0 has the form

$$m_0(\xi) = m_0^1(\xi) + e^{-2\pi i\xi r/N} m_0^2(\xi),$$

for some locally L^2, $1/2$-periodic functions m_0^1 and m_0^2. Define M_0 by

$$M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2.$$

Then, each of the following conditions is necessary and sufficient for the existence of associated wavelets $\psi_1, \ldots, \psi_{2N-1}$:

(a) M_0 *is* $1/4$*-periodic.*
(b) $\sum_{k=0}^{N-1} \delta_{k/2} * \sum_{j\in\mathbb{Z}} \delta_{jN} * |\hat{\phi}|^2 = 1$.
(c) *For any odd integer* m, *we have*

$$\int_{\mathbb{R}} \phi(x)\overline{\phi\left(x - \frac{m}{N}\right)}dx = 0.$$

Gabardo and Nashed[7] provides an extension to the setting of NUMRAs of the standard construction of wavelet analysis which consists in constructing a NUMRA starting from a trigonometric polynomial m_0 of the form

$$m_0(\xi) = m_0^1(\xi) + e^{-2\pi i\xi r/N} m_0^2(\xi), \tag{1.1}$$

where the integers r and N satisfy $N \geq 1$, $1 \leq r \leq 2N-1$, r is odd, r and N are relatively prime and $m_0^1(\xi)$ and $m_0^2(\xi)$ are $1/2$-periodic trigonometric polynomials.

The following theorem by Gabardo and Nashed,[7] generalizes Cohen's result for NUMRA, which gives a sufficient condition for the orthonormality of the collection $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$.

Theorem 1.2. *Let* m_0 *be a trigonometric polynomial of the form "(1.1)" which satisfies* $m_0(0) = 1$ *together with the conditions*

$$\sum_{p=0}^{2N-1} M_0\left(\xi + \frac{p}{4N}\right) = 1, \tag{1.2}$$

$$\sum_{p=0}^{2N-1} \alpha^p M_0\left(\xi + \frac{p}{4N}\right) = 0, \tag{1.3}$$

where $\alpha = e^{-\pi i r/N}$ *and* $M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2$. *Let* ϕ *be defined by the formula*

$$\hat{\phi}(2N\xi) = m_0(\xi)\hat{\phi}(\xi), \tag{1.4}$$

and let $\Lambda = 0, r/N + 2\mathbb{Z}$. *Then a sufficient condition for the collection* $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$ *to be orthonormal in* $L^2(\mathbb{R})$ *is the existence of a constant* $c > 0$ *and of a compact set* $K \subset \mathbb{R}$ *that contains a neighborhood of the origin and satisfies*

$$\sum_{k=0}^{N-1} \delta_{k/2} * \sum_{j\in\mathbb{Z}} \delta_{Nj} * \chi_K = 1,$$

such that

$$\left| m_0\left(\frac{\xi}{(2N)^k}\right)\right| \geq c \quad \forall\, \xi \in K, \quad \forall\, k \geq 1.$$

Furthermore, if the function M_0 *defined by* $M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2$ *is* $1/4$*-periodic, the condition is also necessary.*

This paper is organized as follows. In Sec. 2, we explain certain results of Walsh–Fourier analysis. We present a brief review of generalized Walsh functions, Walsh–Fourier transforms and its various properties, multiresolution p-analysis in $L^2(\mathbb{R}_+)$ introduced by Farkov.[3] Nonuniform multiresolution analysis on positive half-line is defined in Sec. 3 and a necessary and sufficient condition for the existence of associated wavelet is given. In Sec. 4, we construct nonuniform multiresolution analysis on positive half-line starting from a Walsh polynomial m_0 satisfying appropriate conditions and showing that the scaling function φ defined, via the Fourier transform, by the corresponding infinite product

$$\tilde{\varphi}(\xi) = \prod_{k=1}^{\infty} m_0\left(\frac{\xi}{N^k}\right)$$

belong to $L^2(\mathbb{R}_+)$. We construct a nonuniform multiresolution analysis on positive half-line with a compactly supported scaling function φ. In Sec. 5, we find the analogue of Cohen's condition for nonuniform multiresolution on positive half-line which gives necessary and sufficient condition for the orthonormality of the system $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$, $N > 1$ is an integer and r is an odd integer with $1 \le r \le 2N - 1$ such that r and N are relatively prime.

2. Certain Results of Walsh–Fourier Analysis

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{Z}_+ = \{0, 1, \ldots\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}_+$ and for any integer j

$$x_j = [p^j x] \pmod{p}, \quad x_{-j} = [p^{1-j} x] \pmod{p}, \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, \ldots, p-1\}$. It is clear that for each $x \in \mathbb{R}_+$, $\exists\, k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0\ \forall\, j > k$.

Consider on $\mathbb{R}_+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j},$$

with $\xi_j = x_j + y_j \pmod{p}$, $j \in \mathbb{Z}\backslash\{0\}$, where $\xi_j \in \{0, 1, 2, \ldots, p-1\}$ and x_j, y_j are calculated by (2.1).

For $p = 2$ $\oplus$ was introduced by N. J. Fine, see Ref. 13. For $x \in \mathbb{R}_+$ and $j \in \mathbb{N}$ we define the numbers $x_j, x_{-j} \in \{0, 1\}$ as follows:

$$x_j = [2^j x] \pmod{2}, \quad x_{-j} = [2^{1-j} x] \pmod{2}, \tag{2.2}$$

where $[\cdot]$ denotes the integral part of $x \in \mathbb{R}_+$. x_j and x_{-j} are the digits of the binary expansion

$$x = \sum_{j<0} x_j 2^{-j-1} + \sum_{j=0} x_j 2^{-j}. \tag{2.3}$$

(For dyadic x, we obtain an expansion with finitely many nonzero terms.)

For fixed $x, y \in \mathbb{R}_+$, we set

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j},$$

where x_j, y_j are defined in (2.2). By definition $x \ominus y = x \oplus y$ (because $x \oplus x = 0$).

The binary operation $\oplus$ identifies $\mathbb{R}_+$ with the group G_2 (dyadic group with addition modulo two) and is useful in the study of dyadic Hardy classes and image processing, see Refs. 6 and 13.

For $x \in [0,1)$, let $r_0(x)$ is given by

$$r_0(x) = \begin{cases} 1 & x \in [0, 1/p), \\ \varepsilon_p^j & x \in [jp^{-1}, (j+1)p^{-1}), \quad j = 1, 2, \ldots, p-1, \end{cases} \tag{2.4}$$

where $\varepsilon_p = \exp(\frac{2\pi i}{p})$.

The extension of the function r_0 to $\mathbb{R}_+$ is defined by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}_+$. Then the *generalized Walsh functions* $\{w_m(x)\}_{m \in \mathbb{Z}_+}$ are defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j},$$

where

$$m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, \ldots, p-1\},\ \mu_k \neq 0,$$

(the classical Walsh system corresponds for the case $p = 2$).

For $x, w \in \mathbb{R}_+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j) \right), \tag{2.5}$$

where x_j and w_j are given by (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}}, m\right) = w_m\left(\frac{x}{p^{n-1}}\right) \quad \forall\, x \in [0, p^{n-1}), \quad m \in \mathbb{Z}.$$

The *Walsh–Fourier transform* of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\tilde{f}(w) = \int_{\mathbb{R}_+} f(x) \overline{\chi(x, w)} dx, \tag{2.6}$$

where $\chi(x, w)$ is given by (2.5).

If $f \in L^2(\mathbb{R}_+)$ and

$$J_a f(w) = \int_0^a f(x) \overline{\chi(x, w)} dx \quad (a > 0), \tag{2.7}$$

then $\tilde{f}$ is defined as limit of $J_a f$ in $L^2(\mathbb{R}_+)$ as $a \to \infty$.

Properties of Walsh–Fourier transform

If $f \in L^2(\mathbb{R}_+)$, then $\tilde{f} \in L^2(\mathbb{R}_+)$ and

$$\|\tilde{f}\|_{L^2(\mathbb{R}_+)} = \|f\|_{L^2(\mathbb{R}_+)}.$$

If $x, y, w \in \mathbb{R}_+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, w) = \chi(x, w)\chi(y, w), \tag{2.8}$$

see Ref. 6. Thus for fixed x and w, the equality (2.8) holds for all $y \in \mathbb{R}_+$ except for countably many. It is well known that systems $\{\chi(\alpha, \cdot)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0, 1]$.

Let $\{w\}$ denotes the fractional part of w. For any $\varphi \in L^2(\mathbb{R}_+)$ and $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}\int_{\mathbb{R}_+} \varphi(x)\overline{\varphi(x \ominus k)}dx &= \sum_{l=0}^{\infty} \int_l^{l+1} |\tilde{\varphi}(w)|^2 \overline{\chi(k, \{w\})}dw \\ &= \int_0^1 \left(\sum_{l \in \mathbb{Z}_+} |\tilde{\varphi}(w + l)|^2\right) \overline{\chi(k, w)}dw.\end{aligned} \tag{2.9}$$

Therefore, a necessary and sufficient condition for a system $\{\varphi(\cdot \ominus k)/k \in \mathbb{Z}_+\}$ to be orthonormal in $L^2(\mathbb{R}_+)$ is

$$\sum_{l \in \mathbb{Z}_+} |\tilde{\varphi}(w + l)|^2 = 1 \quad \text{a.e.} \tag{2.10}$$

Multiresolution p-analysis in $L^2(\mathbb{R}_+)$ defined by Farkov[3] is as follows.

Definition 2.1. A multiresolution p-analysis in $L^2(\mathbb{R}_+)$ is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$ $(j \in \mathbb{Z})$ such that the following hold:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(ii) $\bigcup V_j$ is dense in $L^2(\mathbb{R}_+)$ and $\bigcap V_j = \{0\}$.
(iii) $f(\cdot) \in V_j \Leftrightarrow f(p\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$.
(iv) $f(\cdot) \in V_0 \Leftrightarrow f(\cdot \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$.
(v) There is a function $\varphi \in L^2(\mathbb{R}_+)$ such that $\{\varphi(\cdot \ominus k)/k \in \mathbb{Z}_+\}$ is an orthonormal basis of V_0.

The function φ is called a scaling function in $L^2(\mathbb{R}_+)$.

Farkov[3] has given a general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}_+)$ arising from scaling filters with p^n many terms. For all integer $p \geq 2$ these wavelets are identified with certain lacunary Walsh series on $\mathbb{R}_+$. In this new setting Farkov has proved the extension of classical results concerning necessary and sufficient condition of wavelets associated with the classical multiresolution analysis.

The following theorem by Farkov generalizes A. Cohen's result:

Theorem 2.1. *Let*

$$m_0(w) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(\alpha, w)}$$

be a polynomial satisfying the following conditions:

(a) $m_0(0) = 1$.
(b) $\sum_{j=0}^{p^n-1} |m_0(sp^{-n} \oplus jp^{-1})|^2 = 1$ *for* $s = 0, 1, \ldots, p^{n-1} - 1$.
(c) *There exists a* W*-compact set* E *such that* $0 \in \mathrm{int}(E)$, $\mu(E) = 1$, $E \equiv [0, 1) \pmod{\mathbb{Z}_+}$ *and*

$$\inf_{j\in\mathbb{N}} \inf_{w\in E} |m_0(p^{-j}w)| > 0.$$

If the Walsh–Fourier transform of $\varphi \in L^2(\mathbb{R}_+)$ *can be written as*

$$\tilde{\varphi}(w) = \prod_{j=1}^{\infty} m_0(p^{-j}w),$$

then φ *is a scaling function in* $L^2(\mathbb{R}_+)$.

3. Nonuniform Multiresolution Analysis on Positive Half-Line

Definition 3.1. For an integer $N > 1$ and an odd integer r with $1 \le r \le 2N - 1$ such that r and N are relatively prime, an associated nonuniform multiresolution analysis on positive half-line is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$, $j \in \mathbb{Z}$ such that the following properties hold:

(i) $V_j \subset V_{j+1}\ \forall\, j \in \mathbb{Z}$.
(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}_+)$.
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(iv) $f(\cdot) \in V_j \iff f(N.) \in V_{j+1}\ \forall\, j \in \mathbb{Z}$.
(v) There exists a function $\varphi \in V_0$ such that $\{\varphi(x \ominus \lambda), \lambda \in \Lambda_+\}$ where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$, is a complete orthonormal system for V_0.

The function φ is called a scaling function in $L^2(\mathbb{R}_+)$.

When $N > 1$, the dilation factor of N ensures that

$$N\Lambda_+ \subset \mathbb{Z}_+ \subset \Lambda_+.$$

By the conditions (iv) and (v) of the Definition 3.1

$$\varphi_{1,\lambda}(x) = N^{1/2}\varphi(Nx \ominus \lambda), \quad \lambda \in \Lambda_+$$

constitute an orthonormal basis in V_0.

Since $V_0 \subset V_1$, the function $\varphi \in V_1$ and has the Fourier expansion

$$\varphi(x) = \sum_{\lambda \in \Lambda_+} h_\lambda (N)^{1/2} \varphi(Nx \ominus \lambda),$$

where $h_\lambda = \int_{\mathbb{R}_+} \varphi(x)\overline{\varphi_{1,\lambda}(x)}dx$.

This implies that

$$\varphi(x) = N \sum_{\lambda \in \Lambda_+} a_\lambda \varphi(Nx \ominus \lambda), \quad \sum_{\lambda \in \Lambda_+} |a_\lambda|^2 < \infty,$$

where $a_\lambda = (N)^{-1/2} h_\lambda$.

Lemma 3.1. *Consider a nonuniform multiresolution analysis on positive half-line as in Definition* 3.1. *Let $\psi_0 = \varphi$ and suppose $\exists\, N-1$ functions ψ_k, $k = 1, 2, \ldots, N-1$ in V_1 such that the family of functions $\{\psi_k(x \ominus \lambda)\}_{\lambda \in \Lambda_+, k=0,1,\ldots,N-1}$ forms an orthonormal system in V_1. Then the system is complete in V_1.*

Proof. Since $\psi_k \in V_1$, $k = 0, 1, \ldots, N-1$ there exists the sequences $\{h_\lambda^k\}_{\lambda \in \Lambda_+}$ satisfying $\sum_{\lambda \in \Lambda_+} |h_\lambda^k|^2 < \infty$ such that

$$\psi_k(x) = \sum_{\lambda \in \Lambda_+} h_\lambda^k N^{1/2} \varphi(Nx \ominus \lambda).$$

This implies that

$$\psi_k(x) = N \sum_{\lambda \in \Lambda_+} a_\lambda^k \varphi(Nx \ominus \lambda), \quad \sum_{\lambda \in \Lambda_+} |a_\lambda^k|^2 < \infty,$$

where $a_\lambda^k = (N)^{-1/2} h_\lambda^k$.

On taking the Walsh–Fourier transform, we have

$$\tilde{\psi}_k(\xi) = m_k\left(\frac{\xi}{N}\right)\tilde{\varphi}\left(\frac{\xi}{N}\right), \tag{3.1}$$

where $m_k(\xi) = \sum_{\lambda \in \lambda_+} a_\lambda^k \overline{\chi(\lambda, \xi)}$ and $\chi(\lambda, \xi) = \exp(\frac{2\pi i}{N} \sum_{j=1}^{\infty}(\lambda_j \xi_{-j} + \lambda_{-j}\xi_j))$ where λ_j and ξ_j are given by (2.1) for a positive integer N.

Since $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$, we can write

$$m_k(\xi) = m_k^1(\xi) + \overline{\chi\left(\frac{r}{N}, \xi\right)} m_k^2(\xi), \quad k = 0, 1, \ldots, N-1, \tag{3.2}$$

where m_k^1 and m_k^2 are locally L^2 functions.

According to Ref. 8 for $\lambda \in \Lambda_+$, where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$ and by assumption we have

$$\begin{aligned}\int_{\mathbb{R}_+} \psi_k(x)\overline{\psi_l(x \ominus \lambda)}dx &= \int_{\mathbb{R}_+} \tilde{\psi}_k(\xi)\overline{\tilde{\psi}_l(\xi)}\,\overline{\chi(\lambda, \xi)}d\xi \\ &= \delta_{kl}\delta_{0\lambda},\end{aligned}$$

where δ_{kl} denotes the Kronecker delta.

Define

$$h_{kl}(\xi) = \sum_{j\in\mathbb{Z}_+} \tilde{\psi}_k(\xi + Nj)\overline{\tilde{\psi}_l(\xi + Nj)}, \quad 0 \le k, \quad l \le N-1.$$

If $\lambda \in \mathbb{Z}_+$, we have

$$\begin{aligned}\int_{\mathbb{R}_+} \psi_k(x)\overline{\psi_l(x\ominus\lambda)}dx &= \int_{[0,N]} h_{kl}(\xi)\overline{\chi(\lambda,\xi)}d\xi \\ &= \int_{[0,1]} \left[\sum_{p=0}^{N-1} h_{kl}(\xi+p)\right]\overline{\chi(\lambda,\xi)}d\xi\end{aligned}$$

On taking $\lambda = \frac{r}{N} + n$ where $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+} \psi_k(x)\overline{\psi_l(x\ominus\lambda)}dx &= \int_{[0,N]} \overline{\chi\left(n+\frac{r}{N},\xi\right)}h_{kl}(\xi)d\xi \\ &= \int_{[0,N]} \overline{\chi(n,\xi)}\,\overline{\chi\left(\frac{r}{N},\xi\right)}h_{kl}(\xi)d\xi \\ &= \int_{[0,N]} \overline{\chi(n,\xi)}\,\overline{\chi\left(\frac{r}{N},\xi\right)}\left[\sum_{p=0}^{N-1} \overline{\chi\left(\frac{r}{N},p\right)}h_{kl}(\xi+p)\right]\, d\xi.\end{aligned}$$

By the orthonormality of the system $\{\psi_k(x\ominus\lambda)\}_{\lambda\in\Lambda_+,k=0,1,\dots,N-1}$ we conclude that

$$\sum_{p=0}^{N-1} h_{kl}(\xi+p) = \delta_{kl} \tag{3.3}$$

and

$$\sum_{p=0}^{N-1} \overline{\chi\left(\frac{r}{N},p\right)}h_{kl}(\xi+p) = 0 \quad \text{i.e. } \sum_{p=0}^{N-1} \alpha^p h_{kl}(\xi+p) = 0, \tag{3.4}$$

where $\alpha = \overline{\chi(r/N,1)}$, since $\overline{\chi(r/N,p)} = [\overline{\chi(r/N,1)}]^p$ for $p = 0,1,\dots,N-1$.

Now we will express the conditions (3.3) and (3.4) in terms of m_k as follows:

$$\begin{aligned}h_{kl}(N\xi) &= \sum_{j\in\mathbb{Z}_+} \tilde{\psi}_k(N\xi+Nj)\overline{\tilde{\psi}_l(N\xi+Nj)} \\ &= \sum_{j\in\mathbb{Z}_+} \tilde{\psi}_k[N(\xi+j)]\overline{\tilde{\psi}_l[N(\xi+j)]} \\ &= \sum_{j\in\mathbb{Z}_+} m_k(\xi+j)\overline{m_l(\xi+j)}|\tilde{\varphi}(\xi+j)|^2 \\ &= [m_k^1(\xi)\overline{m_l^1(\xi)} + m_k^2(\xi)\overline{m_l^2(\xi)}]\sum_{j\in\mathbb{Z}_+} |\tilde{\varphi}(\xi+j)|^2\end{aligned}$$

$$+\left[m_k^1(\xi)\overline{m_l^2(\xi)}\sum_{j\in\mathbb{Z}_+}\chi\left(\frac{r}{N},\xi+j\right)|\tilde{\varphi}(\xi+j)|^2\right]$$
$$+\left[m_k^2(\xi)\overline{m_l^1(\xi)}\sum_{j\in\mathbb{Z}_+}\overline{\chi\left(\frac{r}{N},\xi+j\right)}|\tilde{\varphi}(\xi+j)|^2\right].$$

Therefore,

$$\begin{aligned}h_{kl}(N\xi) &= [m_k^1(\xi)\overline{m_l^1(\xi)}+m_k^2(\xi)\overline{m_l^2(\xi)}]\sum_{j=0}^{N-1}h_{00}(\xi+j)\\ &\quad+\left[m_k^1(\xi)\overline{m_l^2(\xi)}\chi\left(\frac{r}{N},\xi\right)\sum_{j=0}^{N-1}\chi\left(\frac{r}{N},j\right)h_{00}(\xi+j)\right]\\ &\quad+\left[m_k^2(\xi)\overline{m_l^1(\xi)}\overline{\chi\left(\frac{r}{N},\xi\right)}\sum_{j=0}^{N-1}\overline{\chi\left(\frac{r}{N},j\right)}h_{00}(\xi+j)\right]\\ &= m_k^1(\xi)\overline{m_l^1(\xi)}+m_k^2(\xi)\overline{m_l^2(\xi)}.\end{aligned}$$

By using the last identity and Eqs. (3.3) and (3.4), we obtain

$$\sum_{p=0}^{N-1}\left[m_k^1\left(\frac{\xi+p}{N}\right)\overline{m_l^1\left(\frac{\xi+p}{N}\right)}+m_k^2\left(\frac{\xi+p}{N}\right)\overline{m_l^2\left(\frac{\xi+p}{N}\right)}\right]=\delta_{kl}, \tag{3.5}$$

and

$$\sum_{p=0}^{N-1}\alpha^p\left[m_k^1\left(\frac{\xi+p}{N}\right)\overline{m_l^1\left(\frac{\xi+p}{N}\right)}+m_k^2\left(\frac{\xi+p}{N}\right)\overline{m_l^2\left(\frac{\xi+p}{N}\right)}\right]=0, \tag{3.6}$$

$0\le k,l\le N-1$.

Both of these conditions together are equivalent to the orthonormality of the system $\{\psi_k(x\ominus\lambda)\}_{\lambda\in\Lambda_+,k=0,1,\ldots,N-1}$.

The completeness of the system $\{\psi_k(x\ominus\lambda)\}_{\lambda\in\Lambda_+,k=0,1,\ldots,N-1}$ in V_1 is equivalent to the completeness of the system $\{\frac{1}{N}\psi_k(\frac{x}{N}\ominus\lambda)\}_{\lambda\in\Lambda_+,k=0,1,\ldots,N-1}\}$ in V_0. For a given arbitrary function $f\in V_0$, by assumption $\exists$ a unique function $m(\xi)$ of the form $\sum_{\lambda\in\Lambda_+}b_\lambda\overline{\chi(\lambda,\xi)}$ where $\sum_{\lambda\in\Lambda_+}|b_\lambda|^2<\infty$ such that $\tilde{f}(\xi)=m(\xi)\tilde{\varphi}(\xi)$.

Hence, in order to prove the claim, it is enough to show that the system of functions

$$S=\{\overline{\chi(N\lambda,\xi)}m_k(\xi)\}_{\lambda\in\Lambda_+,k=0,1,\ldots,N-1}$$

is complete in $L^2[0,1]$.

Let $g \in L^2[0,1]$, therefore $\exists$ locally L^2 functions g_1 and g_2 such that

$$g(\xi) = g_1(\xi) + \overline{\chi\left(\frac{r}{N}, \xi\right)} g_2(\xi).$$

Assuming that g is orthogonal to all functions in S, we then have for any $\lambda \in \Lambda_+$ and $k \in \{0, 1, \ldots, N-1\}$ that

$$\begin{aligned} 0 &= \int_{[0,1]} \overline{\chi(\xi, N\lambda)} m_k(\xi) \overline{g(\xi)} d\xi \\ &= \int_{[0,1]} \overline{\chi(\xi, N\lambda)} [m_k^1(\xi) \overline{g_1(\xi)} + m_k^2(\xi) \overline{g_2(\xi)}] d\xi. \end{aligned} \tag{3.7}$$

Taking $\lambda = m$ where $m \in \mathbb{Z}_+$ and defining

$$w_k(\xi) = m_k^1(\xi) \overline{g_1(\xi)} + m_k^2(\xi) \overline{g_2(\xi)}, \quad k = 0, 1, \ldots, N-1,$$

we obtain

$$\begin{aligned} 0 &= \int_{[0,1]} \overline{\chi(\xi, Nm)} w_k(\xi) d\xi \\ &= \int_{[0, \frac{1}{N}]} \overline{\chi(\xi, Nm)} \sum_{j=0}^{N-1} w_k\left(\xi + \frac{j}{N}\right) d\xi. \end{aligned}$$

Since this equality holds for all $m \in \mathbb{Z}_+$, therefore

$$\sum_{j=0}^{N-1} w_k\left(\xi + \frac{j}{N}\right) = 0 \quad \text{for a.e. } \xi. \tag{3.8}$$

Similarly, on taking $\lambda = m + \frac{r}{N}$ where $m \in \mathbb{Z}_+$, we obtain

$$\begin{aligned} 0 &= \int_{[0,1]} \overline{\chi(\xi, Nm)}\, \overline{\chi(\xi, r)} w_k(\xi) d\xi \\ &= \int_{[0, \frac{1}{N}]} \overline{\chi(\xi, Nm)}\, \overline{\chi(\xi, r)} \sum_{j=0}^{N-1} \alpha^j w_k\left(\xi + \frac{j}{N}\right) d\xi. \end{aligned}$$

Hence we deduce that

$$\sum_{j=0}^{N-1} \alpha^j w_k\left(\xi + \frac{j}{N}\right) = 0 \quad \text{for a.e. } \xi,$$

which proves our claim. $\square$

If $\psi_0, \psi_1, \ldots, \psi_{N-1} \in V_1$ are as in Lemma 3.1, one can obtain from them an orthonormal basis for $L^2(\mathbb{R}_+)$ by following the standard procedure for construction of wavelets from a given MRA.[2,12,17,18] It can be easily checked that for every $m \in \mathbb{Z}$, the collection $\{N^{m/2}\psi_k(N^m x \ominus \lambda)\}_{\lambda \in \Lambda_+, k=0,1,\ldots,N-1}$ is a complete orthonormal system for V_{m+1}.

Given a NUMRA on positive half-line, we denote by W_m the orthogonal complement of V_m in V_{m+1}, $m \in \mathbb{Z}$. It is clear from (i), (ii) and (iii) of Definition 3.1 that

$$L^2(\mathbb{R}_+) = \bigoplus_{m\in\mathbb{Z}} W_m,$$

where $\oplus$ denotes the orthogonal direct sum with the inner product of $L^2(\mathbb{R}_+)$.

From this it follows immediately that the collection $\{N^{m/2}\psi_k(N^m x \ominus \lambda)\}_{\lambda\in\Lambda_+, m\in\mathbb{Z}, k=1,2,\ldots,N-1}$ forms a complete orthonormal system for $L^2(\mathbb{R}_+)$.

Definition 3.2. A collection $\{\psi_k\}_{k=1,2,\ldots,N-1}$ of functions in V_1 will be called a set of wavelets associated with a given nonuniform multiresolution analysis on positive half-line if the family of functions $\{\psi_k(x \ominus \lambda)\}_{k=1,2,\ldots,N-1,\lambda\in\Lambda_+}$ is an orthonormal system for W_0.

The following theorem proves the necessary and sufficient condition for the existence of associated set of wavelets to nonuniform multiresolution analysis on positive half-line.

Theorem 3.1. *Consider a nonuniform multiresolution analysis on a positive half-line with associated parameters N and r, as in Definition* 3.1, *such that the corresponding space V_0 has an orthonormal system of the form $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$, $\tilde{\varphi}$ satisfies*

$$\tilde{\varphi}(\xi) = m_0\left(\frac{\xi}{N}\right)\tilde{\varphi}\left(\frac{\xi}{N}\right), \tag{3.9}$$

and m_0 has the form

$$m_0(\xi) = m_0^1(\xi) + \overline{\chi\left(\frac{r}{N}, \xi\right)} m_0^2(\xi), \tag{3.10}$$

for some locally L^2 functions m_0^1 and m_0^2. M_0 is defined as

$$M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2. \tag{3.11}$$

Then a necessary and sufficient condition for the existence of associated wavelets $\psi_1, \psi_2, \ldots, \psi_{N-1}$ is that M_0 satisfies the identity

$$M_0(\xi + 1) = M_0(\xi). \tag{3.12}$$

Proof. The orthonormality of the collection of functions $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ which satisfies (3.9), implies the following identities as shown in the proof of Lemma 3.1

$$\sum_{p=0}^{N-1}\left[\left|m_0^1\left(\xi + \frac{p}{N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{N}\right)\right|^2\right] = 1$$

and

$$\sum_{p=0}^{N-1}\alpha^p\left[\left|m_0^1\left(\xi + \frac{p}{N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{N}\right)\right|^2\right] = 0,$$

where $\alpha = \overline{\chi(r/N, 1)}$. Similarly if $\{\psi_k\}_{k=1,2,\ldots,N-1}$ is a set of wavelets associated with the given nonuniform multiresolution analysis then it satisfies the relation (3.1) and the orthonormality of the collection $\{\psi_k\}_{k=0,1,\ldots,N-1}$ in V_1 is equivalent to the identities

$$\sum_{p=0}^{N-1}\left[m_k^1\left(\frac{\xi+p}{N}\right)\overline{m_l^1\left(\frac{\xi+p}{N}\right)} + m_k^2\left(\frac{\xi+p}{N}\right)\overline{m_l^2\left(\frac{\xi+p}{N}\right)}\right] = \delta_{kl}, \tag{3.13}$$

and

$$\sum_{p=0}^{N-1}\alpha^p\left[m_k^1\left(\frac{\xi+p}{N}\right)\overline{m_l^1\left(\frac{\xi+p}{N}\right)} + m_k^2\left(\frac{\xi+p}{N}\right)\overline{m_l^2\left(\frac{\xi+p}{N}\right)}\right] = 0, \tag{3.14}$$

$0 \le k, l \le N-1$.

If $\xi \in [0, 1/N]$ is fixed and $a_k(j) = m_k^1(\xi + j/N)$, $b_k(j) = m_k^2(\xi + j/N)$ are vectors in C^N for $j = 0, 1, \ldots, N-1$ where $0 \le k \le N-1$, then by Lemma 3.5,[6] the solvability of system of Eqs. (3.13) and (3.14) is equivalent to

$$M_0\left(\xi + \frac{j+N}{N}\right) = M_0\left(\xi + \frac{j}{N}\right), \quad \xi \in \left[0, \frac{1}{N}\right], \quad j = 0, 1, \ldots, N-1,$$

which is equivalent to (3.12). □

The main purpose in the remaining sections is to construct a nonuniform multiresolution analysis on positive half-line starting from a Walsh polynomial m_0 satisfying appropriate conditions and finding suitable analogue of Cohen's conditions.

4. Construction of Nonuniform Multiresolution Analysis on Positive Half-Line

Our goal in this section is to construct a nonuniform multiresolution analysis on a positive half-line starting from a polynomial m_0 of the form

$$m_0(\xi) = m_0^1(\xi) + \overline{\chi\left(\frac{r}{N}, \xi\right)}m_0^2(\xi), \tag{4.1}$$

where $N > 1$ is an integer and r is an odd integer with $1 \le r \le 2N-1$ such that r and N are relatively prime and $m_0^1(\xi)$ and $m_0^2(\xi)$ are locally L^2 Walsh polynomials. The scaling function φ associated with given nonuniform multiresolution analysis on positive half-line should satisfy the scaling relation

$$\tilde{\varphi}(\xi) = m_0\left(\frac{\xi}{N}\right)\tilde{\varphi}\left(\frac{\xi}{N}\right). \tag{4.2}$$

Define

$$M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2,$$

and suppose

$$\sum_{p=0}^{N-1} M_0\left(\xi + \frac{p}{N}\right) = 1 \tag{4.3}$$

and

$$\sum_{p=0}^{N-1} \alpha^p M_0\left(\xi + \frac{p}{N}\right) = 0, \tag{4.4}$$

where $\alpha = \overline{\chi(r/N, 1)}$. It follows from (4.2) that

$$\tilde{\varphi}(\xi) = \prod_{k=1}^{\infty} m_0\left(\frac{\xi}{N^k}\right). \tag{4.5}$$

Also assume that $m_0(0) = 1$ in order for the infinite product $\prod_{k=1}^{\infty} m_0(\frac{\xi}{N^k})$ to converge pointwise. For an arbitrary function m_0 of the form (4.1) the conditions (4.3) and (4.4) imply that $|m_0| \leq 1$ a.e. Since if $|m_0(\xi)| > 1$ for a fixed ξ, then $|m_0^1(\xi)| + |m_0^2(\xi)| > 1$ and thus $|M_0(\xi)| > \frac{1}{2}$.

We obtain the inequalities

$$\sum_{p=1}^{N-1} M_0\left(\xi + \frac{p}{N}\right) < \frac{1}{2}$$

and

$$\left|\sum_{p=1}^{N-1} M_0\left(\xi + \frac{p}{N}\right)\right| = |M_0(\xi)| > \frac{1}{2}$$

which yields to a contradiction.

Theorem 4.1. *Let m_0 be a polynomial of the form* (4.1) *where m_0^1 and m_0^2 are locally square integrable functions and M_0 satisfy* (4.3) *and* (4.4). *Let φ be defined by* (4.5) *and assume that the infinite product defining $\tilde{\varphi}$ converges a.e. on $\mathbb{R}_+$. Then the function $\varphi \in L^2(\mathbb{R}_+)$.*

Proof. Consider the integrals

$$A_1 = \int_0^N M_0\left(\frac{\xi}{N}\right) d\xi$$

and

$$A_k = \int_0^{N^k} M_0\left(\frac{\xi}{N^k}\right) \prod_{j=1}^{k-1} \left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi \quad \text{for } k \geq 2,$$

$$A_1 = \int_0^N M_0\left(\frac{\xi}{N}\right) d\xi = \int_0^1 \sum_{p=0}^{N-1} M_0\left(\frac{\xi}{N} + \frac{p}{N}\right) d\xi = 1$$

and

$$A_k = \int_0^{N^k} M_0\left(\frac{\xi}{N^k}\right)\prod_{j=1}^{k-1}\left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi$$

$$= \int_0^{N^{k-1}} \sum_{p=0}^{N-1}\left|m_0\left(\frac{\xi}{N^{k-1}}+p\right)\right|^2 M_0\left(\frac{\xi}{N^k}+\frac{p}{N}\right)\prod_{j=1}^{k-2}\left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi.$$

We find that

$$\begin{aligned}
&\sum_{p=0}^{N-1}\left|m_0\left(\frac{\xi}{N^k}+p\right)\right|^2 M_0\left(\frac{\xi}{N^k}+\frac{p}{N}\right)\\
&\quad=\left[\left|m_0^1\left(\frac{\xi}{N^{k-1}}\right)\right|^2+\left|m_0^2\left(\frac{\xi}{N^{k-1}}\right)\right|^2\right]\sum_{p=0}^{N-1} M_0\left(\frac{\xi}{N^k}+\frac{p}{N}\right)\\
&\qquad+\overline{m_0^1\left(\frac{\xi}{N^{k-1}}\right)}m_0^2\left(\frac{\xi}{N^{k-1}}\right)\chi\left(\frac{r}{N},\frac{\xi}{N^{k-1}}\right)\sum_{p=0}^{N-1}\alpha^p M_0\left(\frac{\xi}{N^k}+\frac{p}{N}\right)\\
&\qquad+m_0^1\left(\frac{\xi}{N^{k-1}}\right)\overline{m_0^2\left(\frac{\xi}{N^{k-1}}\right)}\,\overline{\chi\left(\frac{r}{N},\frac{\xi}{N^{k-1}}\right)}\sum_{p=0}^{N-1}\alpha^{-p} M_0\left(\frac{\xi}{N^k}+\frac{p}{N}\right)\\
&\quad=\left|m_0^1\left(\frac{\xi}{N^{k-1}}\right)\right|^2+\left|m_0^2\left(\frac{\xi}{N^{k-1}}\right)\right|^2\\
&\quad=M_0\left(\frac{\xi}{N^{k-1}}\right).
\end{aligned}$$

This shows that

$$A_k = \int_0^{N^{k-1}} M_0\left(\frac{\xi}{N^{k-1}}\right)\prod_{j=1}^{k-2}\left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi = A_{k-1}.$$

It follows that, for all k

$$A_k = A_{k-1} = A_{k-2} = \cdots = A_1 = 1.$$

Hence

$$\begin{aligned}
\int_0^{N^k}|\tilde{\varphi}(\xi)|^2\,d\xi &\le \int_0^{N^k}\prod_{j=1}^{k-1}\left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi\\
&\le \int_0^{N^k} 2M_0\left(\frac{\xi}{N^k}\right)\prod_{j=1}^{k-1}\left|m_0\left(\frac{\xi}{N^j}\right)\right|^2 d\xi\\
&= 2A_k = 2.
\end{aligned}$$

Since k is arbitrary, it follows that $\varphi \in L^2(\mathbb{R}_+)$. $\square$

We will construct nonuniform multiresolution analysis on a positive half-line from a polynomial m_0 of the form (4.1) which satisfies (4.3) and (4.4) and also the condition $m_0(0) = 1$.

By Theorem 4.1 we obtain a compactly supported function $\varphi \in L^2(\mathbb{R}_+)$,[2] which satisfies

$$\tilde{\varphi}(N\xi) = m_0(\xi)\tilde{\varphi}(\xi). \tag{4.6}$$

Now, it is necessary to determine the orthonormality of the system of functions $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ in $L^2(\mathbb{R}_+)$ where $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$.

If the orthonormality condition is satisfied, we can define

$$V_0 = \overline{\text{span}}\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$$

and V_j for $j \in \mathbb{Z}$ is defined as

$$f(x) \in V_j \Leftrightarrow f\left(\frac{x}{N^j}\right) \in V_0 \tag{4.7}$$

so that (iv) and (v) of Definition 3.1 hold.

Also Eq. (4.6) implies that (i) also holds. The remaining two conditions (ii) and (iii) follow from the results of Theorems 4.2 and 4.3 which are analogies of the results in standard theory.[2,12]

For an integer m, let $\varepsilon_m(\mathbb{R}_+)$ denotes the collection of all functions f on $\mathbb{R}_+$ which are constant on $[sN^{-m}, (s+1)N^{-m})$ for each $s \in \mathbb{Z}_+$. Further we set,

$$\tilde{\varepsilon_m}(\mathbb{R}_+) = \{f : f \text{ is W-continuous and } \tilde{f} \in \varepsilon_m(\mathbb{R}_+)\}$$

and

$$\varepsilon(\mathbb{R}_+) = \bigcup_{m=1}^{\infty} \varepsilon_m(\mathbb{R}_+), \quad \tilde{\varepsilon}(\mathbb{R}_+) = \bigcup_{m=1}^{\infty} \tilde{\varepsilon_m}(\mathbb{R}_+).$$

The following properties are true:

(1) $\varepsilon_m(\mathbb{R}_+)$ and $\tilde{\varepsilon_m}(\mathbb{R}_+)$ are dense in $L^q(\mathbb{R}_+)$ for $1 \le q \le \infty$.
(2) If $f \in L^1(\mathbb{R}_+) \cup \varepsilon_m(\mathbb{R}_+)$, then supp $\tilde{f} \subset [0, N^m]$.
(3) If $f \in L^1(\mathbb{R}_+) \cup \tilde{\varepsilon_m}(\mathbb{R}_+)$, then supp $f \subset [0, N^m]$.

For $\varphi \in L^2(\mathbb{R}_+)$, we put

$$\varphi_{j,\lambda}(x) = N^{j/2}\varphi(N^j x \ominus \lambda), \quad j \in \mathbb{Z}, \ \ \lambda \in \Lambda_+.$$

Let P_j be the orthogonal projection of $L^2(\mathbb{R}_+)$ to V_j.

Theorem 4.2. *Let $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$. Suppose that $\varphi \in L^2(\mathbb{R}_+)$ is such that the collection $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ is an orthonormal system in $L^2(\mathbb{R}_+)$ with closed linear span V_0 and V_j is defined by* (4.7) *then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

Proof. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Given an $\epsilon > 0$ and a continuous function u which is compactly supported in some interval $[0, R]$, $R > 0$ and satisfies $\|f - u\|_2 < \epsilon$. Then we have

$$\|f - P_j u\|_2 \le \|P_j(f - u)\|_2 \le \|f - u\|_2 < \epsilon$$

so that

$$\|f\|_2 < \|P_j u\|_2 + \epsilon.$$

Using the fact that the collection $\{N^{j/2}\varphi(N^j x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ is an orthonormal bases for V_j

$$\begin{aligned}\|P_j u\|_2^2 &= \sum_{\lambda\in\Lambda_+} |\langle P_j u, \varphi_{j,\lambda}\rangle|^2 \\ &= (N)^j \sum_{\lambda\in\Lambda_+} \left|\int_0^R u(x)\overline{\varphi(N^j x \ominus \lambda)}dx\right|^2 \\ &\le (N)^j \|u\|_\infty^2 R \sum_{\lambda\in\Lambda_+} \int_0^R |\varphi(N^j x \ominus \lambda)|^2\, dx,\end{aligned}$$

where $\|u\|_\infty$ denotes the supremum norm of u. If j is chosen small enough so that $RN^j \le 1$, then

$$\begin{aligned}\|P_j u\|_2^2 &\le \|u\|_\infty^2 \int_{S_{R,j}} |\varphi(x)|^2\, dx \\ &= \|u\|_\infty^2 \int_{\mathbb{R}_+} I_{S_{r,j}}(x)|\varphi(x)|^2\, dx, \qquad (4.8)\end{aligned}$$

where $S_{R,j} = \bigcup_{\lambda\in\Lambda_+}\{y \ominus \lambda / y \in [0, RN^j]\}$ and $I_{S_{r,j}}$ denotes the characteristic function of $S_{r,j}$.

It can be easily checked that

$$\lim_{j\to-\infty} I_{S_{R,j}}(x) = 0 \quad \forall\, x \notin \Lambda_+.$$

Thus from Eq. (4.8) by using the dominated convergence theorem, we get

$$\lim_{j\to-\infty} \|P_j u\|_2 = 0.$$

Therefore, we conclude that $\|f\|_2 < \epsilon$ and since ϵ is arbitrary, $f = 0$ and thus $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. $\square$

Theorem 4.3. *Let $\varphi \in L^2(\mathbb{R}_+)$ is such that the collection $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ is an orthonormal system in $L^2(\mathbb{R}_+)$ with closed linear span V_0 and V_j is defined by* (4.7) *and assume that $\tilde{\varphi}(\xi)$ is bounded for all ξ and continuous near $\xi = 0$ with $|\tilde{\varphi}(0)| = 1$, then $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}_+)$.*

Proof. Let $f \in (\bigcup_{j\in\mathbb{Z}} V_j)^{\perp}$.

Given an $\epsilon > 0$ we choose $u \in L^1(\mathbb{R}_+) \cap \varepsilon(\mathbb{R}_+)$ such that $\|f - u\|_2 < \epsilon$.

Then for any $j \in \mathbb{Z}_+$, we have

$$\|P_j f\|_2^2 = \langle P_j f, P_j f\rangle = \langle f, P_j f\rangle = 0$$

and so

$$\|P_j u\|_2 = \|P_j(f-u)\|_2 \le \|f-u\|_2 < \epsilon.$$

Then we put $g(\xi) = \tilde{u}(\xi)\overline{\tilde{\varphi}(N^{-j}\xi)}$ for some function g of the form

$$g(\xi) = g_1(\xi) + \overline{\chi\left(\xi, \frac{r}{N}\right)} g_2(\xi),$$

where g_1 and g_2 are locally square integrable, 1/2-periodic functions.

If $g(\xi)$ has the expansion of the form $\sum_{\lambda\in\Lambda_+} c_\lambda \overline{\chi(\xi,\lambda)}$ on the set $[0,1]$, then

$$\begin{aligned} c_\lambda &= \int_0^1 g(\xi)\chi(\xi,\lambda)d\xi \\ &= \int_{\mathbb{R}_+} \tilde{u}(\xi)\overline{\tilde{\varphi}(N^{-j}\xi)}\chi(\xi,\lambda)d\xi, \quad \lambda\in\Lambda_+. \end{aligned}$$

If $\lambda \in \mathbb{Z}_+$, we have

$$\int_0^1 2g_1(\xi)\chi(\xi,\lambda)d\xi = \int_0^1 \sum_{k\in\mathbb{Z}_+} \tilde{u}(\xi+k)\overline{\tilde{\varphi}\left(\frac{\xi}{N^j} + \frac{k}{N^j}\right)}\chi(\xi,\lambda)d\xi.$$

Therefore

$$g_1(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}_+} \tilde{u}(\xi+k)\overline{\tilde{\varphi}\left(\frac{\xi}{N^j} + \frac{k}{N^j}\right)}.$$

On taking $\lambda = \frac{r}{N} + m$ where $m \in \mathbb{Z}_+$, we obtain

$$g_2(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}_+} \tilde{u}(\xi+k)\overline{\tilde{\varphi}\left(\frac{\xi}{N^j} + \frac{k}{N^j}\right)}\,\overline{\chi\left(\xi+k, \frac{r}{N}\right)}.$$

Therefore

$$g(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}_+} \tilde{u}(\xi+k)\overline{\tilde{\varphi}\left(\frac{\xi}{N^j} + \frac{k}{N^j}\right)}(1+\alpha^k),$$

where $\alpha = \overline{\chi(r/N, 1)}$.

Since the collection $\{N^{j/2}\varphi(N^j x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ is an orthonormal basis for V_j, if in addition $\tilde{u}$ has compact support, for large values of j

$$\|P_j u\|_2^2 = \sum_{\lambda\in\Lambda_+} |\langle u, \varphi_{j,\lambda}\rangle|^2 = \int_{\mathbb{R}_+} |u(\xi)|^2 |\tilde{\varphi}(N^{-j}\xi)|^2 \, d\xi.$$

By Lebesgue dominated convergence theorem as $j \to \infty$, the expression on R.H.S. converges to $|\tilde{\varphi}(0)|^2 \|\tilde{u}\|_2^2$. Therefore

$$\epsilon > \|P_j u\|_2 = \|\tilde{u}\|_2 = \|u\|.$$

Consequently

$$\|f\|_2 < \epsilon + \|u\|_2 < 2\epsilon.$$

Since ϵ is arbitrary, therefore $f = 0$. □

5. The Analogue of Cohen's Condition

On the basis of construction in the previous section it is necessary to determine the orthonormality of the system of functions $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$. We are going to consider the analogue of Cohen's condition for nonuniform multiresolution analysis on positive half-line.

Theorem 5.1. *Let m_0 be a polynomial of the form* (4.1) *which satisfies $m_0(0) = 1$ together with the conditions* (4.3) *and* (4.4). *Let φ be defined by the formula* (4.5) *and $\Lambda_+ = \{0, r/N\} + \mathbb{Z}_+$. Then the following are equivalent*:

(1) *$\exists$ a W-compact set E such that $0 \in \operatorname{int}(E)$, $\mu(E) = 1$, $E \equiv [0,1] (\operatorname{mod} \mathbb{Z}_+)$ and*

$$\inf_{j\in\mathbb{N}} \inf_{w\in E} |m_0(N^{-j}w)| > 0 \tag{5.1}$$

(2) *The system $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_+}$ is orthonormal in $L^2(\mathbb{R}_+)$.*

Proof. We will start by proving (1) $\Rightarrow$ (2).

For $k \in \mathbb{N}$, define

$$\mu_k(\xi) = \prod_{j=1}^{k} m_0\left(\frac{\xi}{N^j}\right) I_E\left(\frac{\xi}{N^j}\right), \quad \xi \in \mathbb{R}_+.$$

Since $0 \in \operatorname{int}(E)$

$$\mu_k \to \tilde{\varphi} \quad \text{pointwise as } k \to \infty. \tag{5.2}$$

By Eq. (4.5), there is a constant $c_1 > 0$ such that

$$\left| m_0\left(\frac{\xi}{N^k}\right)\right| \geq c_1 > 0 \quad \text{for } k \in \mathbb{N},\ \xi \in E.$$

By (a) and (c) $\exists$ an integer k_0 such that

$$m_0\left(\frac{\xi}{N^k}\right) = 1 \quad \text{for } k > k_0,\ \xi \in E.$$

Thus

$$\tilde{\varphi}(\xi) = \prod_{k=1}^{k_0} m_0\left(\frac{\xi}{N^k}\right), \quad \xi \in E$$

and so

$$c_1^{-k_0}|\varphi(\xi)| \ge I_E(\xi), \quad \xi \in \mathbb{R}_+. \tag{5.3}$$

Therefore

$$\begin{aligned}|\mu_k(\xi)| &= \prod_{j=1}^{k}\left|m_0\left(\frac{\xi}{N^j}\right)\right| I_E\left(\frac{\xi}{N^j}\right)\\ &\le c_1^{-k_0}\prod_{j=1}^{k}\left|m_0\left(\frac{\xi}{N^j}\right)\right|\left|\tilde{\varphi}\left(\frac{\xi}{N^j}\right)\right|\end{aligned}$$

which yields

$$|\mu_k(\xi)| \le c_1^{-k_0}|\tilde{\varphi}(\xi)|, \quad k \in \mathbb{N},\ \ \xi \in \mathbb{R}_+.$$

This implies

$$|\mu_k(\xi)| \le c|\tilde{\varphi}(\xi)|, \quad k \in \mathbb{N},\ \ \xi \in \mathbb{R}_+.$$

Therefore by Lebesgue DCT the sequence $\{\mu_k\}$ converges to $\tilde{\varphi}$ in $L^2(\mathbb{R}_+)$ norm.

We will compute the integral

$$\int_{\mathbb{R}_+} |\mu_k(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi \quad \text{where } \lambda,\sigma \in \Lambda_+.$$

If $k = 1$, then

$$\begin{aligned}\int_{\mathbb{R}_+} |\mu_1(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi &= \int_{\mathbb{R}_+}\left|m_0\left(\frac{\xi}{N}\right)\right|^2 I_E\left(\frac{\xi}{N}\right)\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi\\ &= N\int_E |m_0(\xi)|^2\overline{\chi(\lambda\ominus\sigma,N\xi)}d\xi.\end{aligned}$$

Using the assumption $E \equiv [0,1] \pmod{\Lambda_+}$, we get

$$\begin{aligned}\int_{\mathbb{R}_+} |\mu_1(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi &= N\int_{[0,1]} |m_0(\xi)|^2\overline{\chi(\lambda\ominus\sigma,N\xi)}d\xi\\ &= N\int_{[0,1]} M_0(\xi)\overline{\chi(\lambda\ominus\sigma,N\xi)}d\xi\\ &= N\int_{[0,1/N]}\left[\sum_{p=0}^{N-1} M_0\left(\xi+\frac{p}{N}\right)\overline{\chi(\lambda\ominus\sigma,p)}\right]\\ &\quad\times\overline{\chi(\lambda\ominus\sigma,N\xi)}d\xi\end{aligned}$$

$$= N \int_{[0,1/N]} \overline{\chi(\lambda \ominus \sigma, N\xi)} d\xi$$

$$= \int_{[0,1]} \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi = \delta_{\lambda\sigma}.$$

For $k > 1$

$$\int_{\mathbb{R}_+} |\mu_k(\xi)|^2 \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi = \int_{\mathbb{R}_+} \left[\prod_{j=1}^{k} \left| m_0 \left(\frac{\xi}{N^j} \right) \right|^2 \right] I_E \left(\frac{\xi}{N^k} \right) \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi.$$

Setting $E_k = \{\xi \in \mathbb{R}_+ / N^{-k}\xi \in E\}$ and $w = N^{-k}\xi$, we have

$$\int_{\mathbb{R}_+} |\mu_k(\xi)|^2 \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi = \int_{E_k} \prod_{j=1}^{k} \left| m_0 \left(\frac{\xi}{N^j} \right) \right|^2 \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi$$

$$= N^k \int_E |m_0(w)|^2 \prod_{j=1}^{k-1} |m_0(N^j w)|^2 \overline{\chi(\lambda \ominus \sigma, N^k w)} dw.$$

Using the assumption $E \equiv [0,1] \pmod{\mathbb{Z}_+}$ and changing the variable, we get

$$\int_{\mathbb{R}_+} |\mu_k(\xi)|^2 \overline{\chi(\lambda \ominus \sigma, \xi)} d\xi$$

$$= N^k \int_0^1 \prod_{j=0}^{k-1} |m_0(N^j \xi)|^2 \overline{\chi(\lambda \ominus \sigma, N^k \xi)} d\xi$$

$$= N^k \int_0^1 \left[\prod_{j=1}^{k-1} |m_0(N^j \xi)|^2 \right] M_0(\xi) \overline{\chi(\lambda \ominus \sigma, N^k \xi)} d\xi$$

$$= N^k \int_{[0,1/N]} \left[\prod_{j=2}^{k-1} |m_0(N^j \xi)|^2 \right] \left[\sum_{p=0}^{N-1} |m_0(N\xi + p)|^2 M_0 \left(\xi + \frac{p}{N} \right) \right]$$

$$\times \overline{\chi(\lambda \ominus \sigma, N^k \xi)} d\xi,$$

where

$$\sum_{p=0}^{N-1} |m_0(N\xi + p)|^2 M_0(\xi + p)$$

$$= \sum_{p=0}^{N-1} |m_0^1(N\xi + p) + \overline{\chi \left(\frac{r}{N}, N\xi + p \right)} m_0^2(N\xi + p)|^2 M_0 \left(\xi + \frac{p}{N} \right)$$

$$= [|m_0^1(N\xi)|^2 + |m_0^2(N\xi)|^2] \sum_{p=0}^{N-1} M_0 \left(\xi + \frac{p}{N} \right)$$

$$+\left[m_0^1(N\xi)\overline{m_0^2(N\xi)\chi\left(\frac{r}{N},N\xi\right)\sum_{p=0}^{N-1}\chi\left(\frac{r}{N},p\right)}M_0\left(\xi+\frac{p}{N}\right)\right]$$

$$+\left[m_0^2(N\xi)\overline{m_0^1(N\xi)}\chi\left(\frac{r}{N},N\xi\right)\sum_{p=0}^{N-1}\chi\left(\frac{r}{N},p\right)M_0\left(\xi+\frac{p}{N}\right)\right]$$

$$=|m_0^1(N\xi)|^2+|m_0^2(N\xi)|^2$$

$$=M_0(N\xi).$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}_+}&|\mu_k(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi\\ &=N^k\int_{[0,1/N]}\left[\prod_{j=2}^{k-1}|m_0(N^j\xi)|^2\right]M_0(N\xi)\overline{\chi(\lambda\ominus\sigma,N^k\xi)}d\xi\\ &=N^{k-1}\int_{[0,1]}\left[\prod_{j=2}^{k-1}|m_0(N^{j-1}\xi)|^2\right]M_0(\xi)\overline{\chi(\lambda\ominus\sigma,N^{k-1}\xi)}d\xi\\ &=N^{k-1}\int_{[0,1]}\prod_{j=1}^{k-2}|m_0(N^j\xi)|^2M_0(\xi)\overline{\chi(\lambda\ominus\sigma,N^{k-1}\xi)}d\xi\\ &=\int_{\mathbb{R}_+}|\mu_{k-1}(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi.\end{aligned}$$

Therefore for any $k\in\mathbb{N}$

$$\int_{\mathbb{R}_+}|\mu_k(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi=\delta_{\lambda\sigma},\quad \lambda,\sigma\in\Lambda_+.$$

In particular for $k\in\mathbb{N}$

$$\int_{\mathbb{R}_+}|\mu_k(\xi)|^2\,d\xi=1.$$

Since $\mu_k\to\tilde{\varphi}$ pointwise as $k\to\infty$ by using Fatou's lemma, we obtain

$$\int_{\mathbb{R}_+}|\tilde{\varphi}(\xi)|^2\,d\xi\le 1.$$

From Eqs. (5.2) and (5.3), by Lebesgue's DCT it follows that

$$\int_{\mathbb{R}_+}|\tilde{\varphi}(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi=\lim_{k\to\infty}\int_{\mathbb{R}_+}|\mu_k(\xi)|^2\overline{\chi(\lambda\ominus\sigma,\xi)}d\xi.$$

Therefore,

$$\int_{\mathbb{R}_+}\varphi(x\ominus\lambda)\overline{\varphi(x\ominus\sigma)}=\delta_{\lambda\sigma},\quad \lambda,\sigma\in\Lambda_+.$$

We will now prove the converse (2) $\Rightarrow$ (1).

Since the system $\{\varphi(x \ominus \lambda)\}_{\lambda \in \Lambda_+}$ is orthonormal in $L^2(\mathbb{R}_+)$, therefore we have the identity

$$\sum_{j \in \mathbb{Z}_+} |\tilde{\varphi}(\xi + Nj)|^2 = M_0\left(\frac{\xi}{N}\right) \tag{5.4}$$

which can be deduced as in proof of Theorem 3.1.

From Eq. (4.3), we obtain

$$\sum_{p=0}^{N-1} M_0\left(\xi + \frac{p}{N}\right) = 1$$

and thus we can find, for each $\xi \in [0,1]$ an integer $p(\xi) \in \{0, 1, 2, \ldots, N-1\}$ such that

$$M_0\left(\frac{\xi}{N} + \frac{p(\xi)}{N}\right) \geq \frac{1}{N}.$$

Therefore by using (5.4) there exists $l(\xi) \in \mathbb{N}$ such that

$$\sum_{j \leq l(\xi)} |\tilde{\varphi}(\xi + p(\xi) + Nj)|^2 > \frac{1}{2N}.$$

Since $\tilde{\varphi}$ is continuous, the finite sum $\sum_{j \leq l(\xi)} |\tilde{\varphi}(\cdot + p(\xi) + Nj)|^2$ is continuous. Therefore there exists, for every ξ in $[0,1]$, a neighborhood $\{\zeta : |\zeta - \xi| \leq R_\xi\}$ so that, for all ζ in this neighborhood

$$\sum_{j \leq l(\xi)} |\tilde{\varphi}(\zeta + p(\xi) + Nj)|^2 > \frac{1}{2N}.$$

Since $[0,1]$ is compact, we can find a finite subset of the collection of intervals $\{\zeta : |\zeta - \xi| \leq R_\xi\}$ which covers $[0,1]$. Let l_0 be the maximum of the corresponding numbers $l(\xi)$ associated with this finite covering. Then for all $\zeta \in [0,1]$

$$\sum_{j \leq l_0} |\tilde{\varphi}(\zeta + p(\xi) + Nj)|^2 > \frac{1}{2N}.$$

It follows that for every $\xi \in [0,1]$ $\exists$ $p(\xi)$ between 0 and l_0 such that

$$|\tilde{\varphi}(\xi + p(\xi) + Nk(\xi))|^2 > [2N(2l_0 + 1)]^{-1} = C.$$

By continuity of $\tilde{\varphi}$, there exists for every ξ in $[0,1]$, a neighborhood $\{\zeta : |\zeta - \xi| \leq r(\xi)\}$ so that, for all ζ in this neighborhood

$$|\tilde{\varphi}(\zeta + p(\xi) + Nk(\xi))|^2 > C.$$

Let $U(\xi) \equiv \{\zeta \in [0,1], |\zeta - \xi| \leq r(\xi)\}$.

Since $\tilde{\varphi}(0) = 1$, we can take $p(0) = k(0) = 0$.

Since $[0,1]$ is compact, we can find a finite collection $\{U(\xi_i)\}_{i=0}^{M}$ which covers $[0,1]$ with $\xi_0 = 0$.

Define

$$S_0 = U(\xi_0)$$

and

$$S_l = U(\xi_l) \Big\backslash \bigcup_{j=0}^{l-1} U(\xi_j) \quad \text{if } 1 \le l \le M.$$

The S_l, $0 \le l \le M$ form a partition of $[0,1]$.

We define

$$E = \bigcup_{l=0}^{M} \overline{(S_l + p(\xi_l) + Nk(\xi_l))},$$

then E is clearly compact $E \equiv [0,1] \pmod{\mathbb{Z}_+}$.

By construction, $|\tilde{\varphi}(\xi)| \ge C$ on E and E contains a neighborhood of 0.

Next we will show that E satisfies (5.1).

From the remark in [2, p. 183], we need only to check that $\inf_{\xi\in E} |m_0(\frac{\xi}{N^j})| > 0$ for a finite number j, $1 \le j \le j_0$.

For $\xi \in E$, we have

$$|\tilde{\varphi}(\xi)| = \left(\prod_{j=1}^{j_0} \left| m_0\left(\frac{\xi}{N^j}\right)\right|\right) \left|\tilde{\varphi}\left(\frac{\xi}{N^{j_0}}\right)\right| \tag{5.5}$$

is bounded below away from zero.

Since $|\tilde{\varphi}|$ is also bounded, the first factor on the right-hand side of above equation has therefore no zeros on the compact set E. Since finite product of a continuous function is also continuous, therefore

$$\prod_{j=1}^{j_0} \left| m_0\left(\frac{\xi}{N^j}\right)\right| \ge C_1 > 0 \quad \text{for } \xi \in E.$$

Since $|m_0| \le 1$, we therefore have, for any j, $1 \le j \le j_0$

$$\left| m_0\left(\frac{\xi}{N^j}\right)\right| \ge \prod_{j'=1}^{j_0} \left| m_0\left(\frac{\xi}{N^{j'}}\right)\right| \ge C_1 > 0.$$

This proves that (5.1) is satisfied. □

Example 5.1. Let $N = 2$, $r = 3$ and $m_0^1(\xi) = m_0^2(\xi) = \frac{1}{2}$, $\xi \in [0,2)$.

Since $m_0^1(\xi) = m_0^1(\xi) + \overline{\chi(\frac{r}{N},\xi)} m_0^2(\xi)$ and

$$\overline{\chi\left(\frac{3}{2},\xi\right)} = \begin{cases} 1 & \xi \in \left[0, \frac{1}{2}\right) \cup \left[\frac{3}{2}, 2\right) \\ -1 & \xi \in \left[\frac{1}{2}, \frac{3}{2}\right) \end{cases},$$

therefore

$$m_0(\xi) = \begin{cases} 1 & \xi \in \left[0, \dfrac{1}{2}\right) \cup \left[\dfrac{3}{2}, 2\right) \\ 0 & \xi \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right) \end{cases}.$$

Since $M_0(\xi) = |m_0^1(\xi)|^2 + |m_0^2(\xi)|^2$, therefore $M_0(\xi) = \frac{1}{2}$, $\xi \in [0, 2)$ and

$$\sum_{p=0}^{N-1} M_0\left(\xi + \frac{p}{N}\right) = 1, \quad \sum_{p=0}^{N-1} \alpha^p M_0\left(\xi + \frac{p}{N}\right) = 0$$

where $\alpha = \overline{\chi(\frac{3}{2}, 1)} = -1$.

Then from Eq. (4.5), we have $\tilde{\varphi}(\xi) = I_{[0,1)}(\xi)$. By Theorem 5.1 this function generates a nonuniform multiresolution in $L^2(\mathbb{R}_+)$.

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